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REDUCTION OF THE ENDURANCE LIMIT AS A RESULT OF STRESS INTERACTION IN FATIGUE

Robert A. Heller

Columbia University

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WRIGHT AIR DEVELOPMENT DIVISION

NOTICES

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FOREWORD

This report was prepared by the Department of Civil Engineering and Engineering Mechanics of Columbia University under USAF Contract No. AF 33(616)-7042. The contract was initiated under Project No. 7351 "Metallic Materials," Task No. 73521, "Behavior of Metals." The work was administered under the direction of the Materials Central, Directorate of Advanced Systems Technology, Wright Air Development Division, with Mr. D.M. Forney, Jr. acting as project engineer.

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The cooperation and continued interest of Mr. D.M. Forney, Jr. is gratefully acknowledged.

ABSTRACT

This paper presents the results of an investigation of the effects of stress interaction on fatigue life of aircraft structural materials subjected to randomized load spectra. All three materials: 2024 and 7075 aluminum and SAE 4340 steel exhibit fatigue lives shorter than those predicted on the basis of the linear (Miner) damage rule. A quasi-linear rule is proposed with a variable, spectrum dependent, endurance limit producing safe life estimates; the dependence of the endurance limit on the stress spectrum and its resulting design inadequacy is shown.

Tests were performed on high speed, programmed, rotating bending fatigue machines of special design.

PUBLICATION REVIEW

This report has been reviewed and is approved.

FOR THE COMMANDER:

Walter J. Trapp

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

D = Accumulated Fatigue Damage

$h=\left(\frac{1}{s_c-s_o}\right)^{\alpha}$ = Load parameter (slope of exponential spectrum in semilogarithmic scale)

i = Subscript indicating reference to ith stress level

$L(N)$ = Probability of "survival": probability of fatigue life > N

n = Total number of discrete stress levels in spectrum

N = Fatigue life (number of cycles to failure) in general

N_0 = Minimum fatigue life (minimum number of cycles to failure) in general

The above symbols for the various fatigue lives are used with the following subscripts and/or superscripts:

N_s, N_{0s}, or N_{si}, N_{0si}, Refer to fatigue lives at constant stress amplitudes directly observed and used to estimate spectrum fatigue life on the basis of the usual linear damage rule

N_R, N_{0R}, Refer to fatigue lives under randomized (spectrum) loading estimated on the basis of the usual linear damage rule

N'_s, N'_{0s}, N'_{si}, N'_{0si} Refer to interaction fatigue lives at constant stress amplitude

N'_R, N'_{0R}, Refer to fatigue lives under randomized (spectrum) loading directly observed or estimated on the basis of damage interaction rule

p_i = Relative frequency ratio of cycles of stress amplitude S_i in spectrum

$p(s)$= Probability density function

$P(s)$ = Cumulative probability function

$$P(s) = \int_{-\infty}^{s} p(s)ds$$

$P^*(s)$= Complementary probability function = $1 - P(s)$

S, S_i = Constant stress amplitude

$\bar{S}$ = Upper limit of stress interaction phenomenon (estimated)

S_D = Stress producing maximum damage

S_e = Conventional endurance limit

S'_e = Endurance limit in randomized (spectrum) test (estimated)

S_c, S_o = Characteristic and minimum stress levels; parameters of load spectrum

S_m = Maximum stress amplitude used in randomized (spectrum) fatigue test

S_1 = Lowest stress amplitude used in randomized (spectrum) fatigue test

s, s_i, s_o, s'_e,

s_c, $\bar{s}$, s_m = Stress ratio obtained by dividing respective stress by σ_u

Δs = Difference between adjacent test stress ratios $(s_{i+1}-s_i)$

$T(s)$= Return number $1/P^*(s)$

V_s, V_{si}, V'_s, V'_{si},

V_R, V'_R = "Characteristic" values of extreme value distributions of fatigue lives N_s, N_{si}, N'_s, N'_{si}, N_R, N'_R, at $L = 1/e$

z = Random variable

α = Scale parameter of extremal distribution

β = Scale parameter of extremal distribution

$\Gamma(\rho)$ = Gamma function $\int_0^\infty e^{-x} x^{(\rho-1)}\,dx$

$\Gamma_z(\rho)$ = Incomplete gamma function $\int_0^z e^{-x} x^{(\rho-1)}\,dx$

$\Gamma^z(\rho)$ = Complement of incomplete gamma function $\int_z^\infty e^{-x} x^{(\rho-1)}\,dx$

ν = Slope of log $(S-S_e)$-log V, diagram

ρ = Slope of log $(S-S'_e)$-log V'_s diagram

σ_u = Ultimate tensile strength

$\bar{\omega}$ = V/V'_R Average stress interaction factor, reciprocal of sum of cycle ratios

ω_s = Stress interaction factor at S

ω_V = Stress interaction factor at V_s

1. Introduction

Several investigations were conducted in recent years to obtain a reliable method for the interpretation of the fatigue life of structures subjected to spectrum type loading. While the general problem of correlating test results obtained on simple laboratory specimens under simulated loading conditions with the behavior of complex structures is far from being solved, the partial problem of prediction of fatigue life of specimens is now better understood.

It has been shown previously [1] that "life reducing" interaction between frequent low and infrequent high stress amplitudes based on the concept of slip accumulated into striations [2] leads to a quasi-linear damage rule and conservative estimates of fatigue lives [3]. For this purpose "fictitious" interaction $S\text{-}V_s'$ diagrams were constructed from which the shortened fatigue lives were obtained.

The previous interpretation of the random fatigue tests was based on the following simplifications: the endurance limit of the $S\text{-}V_s'$ relation was assumed to be too low to be significant; a high stress level $\bar{S}$ above which fatigue is replaced by alternating plasticity was chosen arbitrarily, and only simple exponential stress spectra were examined.

The purpose of the present paper is to generalize the previous approach using the full interaction damage rule by the consideration of a variable endurance limit in conjunction with a constant-slope $S\text{-}V_s'$ relation, the elimination of $\bar{S}$, and the inclusion of generalized (skewed) exponential stress spectra and additional test data.

Three aircraft structural materials, 2024 and 7075 aluminum and SAE 4340 steel, were investigated using specially designed [4] rotating bending fatigue machines on which up to seven load levels, controlled by a programmed tape, may be applied to the specimen. The specimens used were 5/16 in. dia. bars with a central section 1 in. long that is gradually reduced to 3/16 in. dia. Table 1 lists the physical properties of the three materials.

2. Summary of Cumulative Damage Theory

The cumulative damage theory presented earlier [1] assumes that the interaction between infrequent high stress amplitudes and frequent low stress amplitudes of a random spectrum produces initiation or acceleration of damage at the low stress amplitudes disproportionately higher than that predicted on the basis of the constant amplitude $S\text{-}V_s$ relation. Though observations have shown that the initial application of high stress amplitudes may produce an increased fatigue life at the subsequent low stress amplitudes due to strain hardening of the material, such results can not be expected in random tests of smooth unnotched specimens essentially free of residual stresses; consequently, only life reducing interaction will be considered. Moreover, little stress interaction should be expected in tests in which

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the proportion of high stress levels is large enough to produce a significant amount of damage on its own so that the test results are governed, essentially, by the high stress levels alone.

It is reasonable to assume that the conventional endurance limit of a material will not remain unaffected if the applied stress spectrum contains stress levels both below and above this limit, because even a non-propagating crack, that would remain static under the application of very low loads, may become active when a few intermittent high loads are applied.

The complex effects of "life reducing" interaction may therefore be represented most effectively by interaction factors, $\omega_s > 1$ that will reduce the characteristic constant amplitude fatigue life at a particular stress level from V_s to $V_s' = V_s/\omega_s$, or $\omega_v > 1$ that will reduce the stress level at a particular fatigue life from S to $S' = S/\omega_v$, both interaction factors being functions of the stress spectrum and related to each other. With their aid an interaction S-V_s' diagram differing in slope and endurance limit from the real S-V_s diagram may be constructed (Fig. 1) and expressed as simple power function of the form

$$\frac{V_s}{V_m} = \left(\frac{s_m - s_e}{s - s_e}\right)^{\nu} \qquad 2.1$$

$$\frac{V_s'}{V_m} = \left(\frac{s_m - s_e'}{s - s_e'}\right)^{\rho} \qquad 2.2$$

where V_s and V_s' are, respectively, the characteristic values (at $L = 1/e$) of the conventional constant amplitude fatigue life and the interaction life, V_m is the conventional constant amplitude fatigue life at the maximum stress level ratio s_m of the spectrum, s the test stress amplitude ratio, s_e and s_e' the conventional and the reduced endurance limit ratios, ν and ρ the slopes of the two lines, where $\nu > \rho$; the stress ratio is defined as the ratio of the test stress to the ultimate tensile strength in tension $s = S/\sigma_u$. The two equations related through the interaction factors may be expressed as $V_s = \omega_s V_s'$ and $s = \omega_v \cdot s'$ or from Eq. 2.1 and 2.2

$$\omega_s = \left(\frac{s_m - s_e}{s - s_e}\right)^{\nu} \Big/ \left(\frac{s_m - s_e'}{s - s_e'}\right)^{\rho} \qquad 2.3$$

$$\omega_v = \frac{s}{(s_m - s_e')\left(\frac{s_m - s_e}{s - s_e}\right)^{-\nu/\rho} + s_e'} = \frac{s}{\omega_s^{-1/\rho}\,(s - s_e') + s_e'} \qquad 2.4$$

For the simplified cases where both s_e and s_e' are assumed to be zero Eq. 2.3 and 2.4 reduce to the form

$$\omega_s = \left(\frac{s_m}{s}\right)^{\nu - \rho} \quad \text{and} \quad \omega_s = \omega_v^{\rho} \qquad 2.5$$

For high stress levels, as s approaches s_m both factors approach unity showing that no interaction occurs at the highest stress level of the spectrum, while for stresses approaching s_e, the interaction factor ω_s becomes very large and ω_v approaches $\omega_v = s_e/s_e'$.

With the use of the interaction factors the linear damage rule of Palmgren [5] and Miner

$$V_R = 1/\Sigma(p_i/V_{si}) \qquad 2.6$$

may be modified to produce the observed fatigue life

$$V_R' = 1/\Sigma(p_i\frac{\omega_{si}}{V_{si}}) = 1/\Sigma(p_i/V_{si}') \qquad 2.7$$

where V_R and V_R' are, respectively, the estimated and observed life under a randomized spectrum of stress amplitudes and p_i is the frequency of occurrence of the ith stress level.

Combining Eqs. 2.2 and 2.7 the modified linear damage rule in terms of stresses

$$\frac{V_R'}{V_m} = 1/\Sigma p_i(\frac{s-s_e'}{s_m-s_e'})^\rho \qquad 2.8$$

results.

3. Load Spectra

The determination of a load spectrum representative of actual service conditions on an aircraft which is to be applied to a test specimen has long been a topic of discussion. It has been shown by Lundberg [6] that a simple exponential spectrum will adequately describe gust and maneuver loads on airplane wings, while a recent paper by Weibull [7] expresses sonic noise spectra in terms of extremal (Weibull) distributions. Because of its versatility and simplicity the extremal load distribution was adopted in the present investigation. The frequency distribution has the form

$$p(s) = \frac{\alpha}{s_c-s_o}\left(\frac{s-s_o}{s_c-s_o}\right)^{(\alpha-1)} e^{-\left(\frac{s-s_o}{s_c-s_o}\right)^\alpha}, \qquad 3.1$$

while the cumulative distribution $P(s) = 1-P^*(s)$, where

$$P^*(s) = \int_s^\infty p(s)\,ds = e^{-\left(\frac{s-s_o}{s_c-s_o}\right)^\alpha} \qquad 3.2$$

represents the frequency or probability of values exceeding s; the return number of such values $T(s) = \frac{1}{P^*(s)}$. In the above expressions s is the non-dimensional stress-amplitude ratio, s_0 the lowest limit of expected stress amplitude ratios, s_c the characteristic stress amplitude ratio similar to (mode) of the spectrum at $P^*(s_c) = 1/e$, and α is a parameter. It should be noted that for $\alpha = 1$ Lundberg's simple exponential distribution, $P^* = e^{-h(s-s_0)}$ results, with slope $h = (\frac{1}{s_c-s_o})$ on a semi logarithmic plot. For $\alpha = 2$, Eq. (3.2) is known as the Rayleigh distribution while for $\alpha = 3.57$ a good approximation to the normal distribution[4] results. Weibull has shown [7] that Eq. 3.2 is applicable to spectra containing a mean stress as well as to those with zero mean stress, as is the case in the present investigation. Figure 2 presents some typical stress spectra while Table 2 lists the relevant parameters and $P^*(s)$ of the distributions used in the tests. It is to be noted that distributions A-C" were designed some time ago on the basis of available flight data [8] without a theoretical probability density function in mind; extremal distributions were fitted to the data later and consequently the parameters listed for these distributions are only approximate.

4. Analysis of Damage Accumulation

The inherent scatter of fatigue test results makes it necessary to associate both the conventional and the interaction fatigue diagrams Eq. 2.1 and 2.2 as well as random test results with a particular level of probability of survival. On the basis of extensive investigations [9] the so called Third Asymptotic distribution of extreme (smallest) values limited by a minimum life N_o has been found to reproduce fatigue test data fairly well; consequently, the probability of surviving N stress cycles will in this report be represented by the survivorship function

$$L(N) = e^{-[(N-N_o)/(V-N_o)]^{\beta}} \qquad 4.1$$

a distribution identical with the one used to define load spectra in Eq. 3.2; V the characteristic value at the probability level $L(V) = 1/e$ is close to the mode of the distribution and β is a scale parameter. The same expression is valid for constant amplitude (N_s) and variable amplitude (N_R) tests.

The cumulative damage relation will be developed for the characteristic value of the observed fatigue life V_R' on the basis of the modified linear damage rule Eqs. 2.7 and 2.8 where the summation is replaced by integration and the frequency of occurrence of individual stress amplitudes by the continuously varying frequency distribution function $p(s)$ according to Eq. 3.1.

$$D = \int_{s_1}^{s_m} p(s) V_R'/V_s' \, ds = \frac{V_R'}{V_m} \int_{s_1}^{s_m} \left(\frac{s-s_e'}{s_m-s_e'}\right)^{\rho} \frac{\alpha}{s_c-s_o} \left(\frac{s-s_o}{s_c-s_o}\right)^{(\alpha-1)} e^{-\left(\frac{s-s_o}{s_c-s_o}\right)^{\alpha}} ds = 1 \qquad 4.2$$

The limits of integration s_1 and s_m are the lowest and the highest stress amplitude ratios of the test spectrum. Changing the variable to

$$z = \left(\frac{s-s_o}{s_c-s_o}\right)^{\alpha}$$

the integral can be simplified

$$\frac{V_R'}{V_m} \left(\frac{s_c-s_o}{s_m-s_e'}\right)^{\rho} \int_{z_1}^{z_m} \left[z^{1/\alpha} + \left(\frac{s_o-s_e'}{s_c-s_o}\right)\right]^{\rho} e^{-z} \, dz = 1 \qquad (4.3)$$

expanding the integrand into a binomial series with the abbreviation

$$-(s_e' - s_o)/(s_c - s_o) = -(z_e')^{1/\alpha}$$

Eq. 4.2 can be written in the form

$$\frac{V_R'}{V_m}\left(\frac{s_c-s_o}{s_m-s_e'}\right)^{\rho}\int_{z_1}^{z_m}\left\{z^{\rho/\alpha} - \rho[z_e' z^{(\rho-1)}]^{1/\alpha} + \frac{\rho(\rho-1)}{2!}[(z_e')^2 z^{(\rho-2)}]^{1/\alpha} - \dots\right\}e^{-z}\,dz = 1 \qquad 4.4$$

Integrating term by term and noting that $\int_0^z z^\rho e^{-z}\,dz = \Gamma_z(\rho+1)$ is the incomplete gamma function with upper limit z:

$$\frac{V_R'}{V_m}\left(\frac{s_c - s_o}{s_m - s_e'}\right)^{\rho}\left\{\left[\Gamma_{z_m}\left(\frac{\rho}{\alpha}+1\right) - \Gamma_{z_1}\left(\frac{\rho}{\alpha}+1\right)\right] - \rho z_e'^{1/\alpha}\left[\Gamma_{z_m}\left(\frac{\rho-1}{\alpha}+1\right) - \Gamma_{z_1}\left(\frac{\rho-1}{\alpha}+1\right)\right] + \right.$$

$$\left.\rho\left(\frac{\rho-1}{2!}\right) z_e'^{2/\alpha}\left[\Gamma_{z_m}\left(\frac{\rho-2}{\alpha}+1\right) - \Gamma_{z_1}\left(\frac{\rho-2}{\alpha}+1\right)\right] - \cdots\right\} = 1 \qquad 4.5$$

An analogous expression is obtained for the linear accumulation fatigue life V_R by replacing s_e' with the conventional endurance limit ratio s_e and ρ with ν. The ratio $V_R'/V_R = 1/\bar{\omega}$ is the ordinary cumulative cycle ratio and $\bar{\omega}$ may be denoted as an over all interaction factor, $\bar{\omega} > 1$, for the spectrum. The above transcendental equation is a function of the parameters ρ and s_e' which may be obtained from experiment. Considerable simplification of Eq. 4.2 can be achieved for simple exponential spectra with $\alpha = 1$, $(s_c - s_o) = 1/h$, by substituting $z = h(s - s_e')$ in Eq. 4.2

$$\frac{V_R'}{V_m}\frac{e^{z_o}}{(z_m^\rho)}\int_{z_1}^{z_m} z^\rho e^{-z}\,dz = 1 = \frac{V_R'}{V_m}\frac{e^{z_o}}{(z_m)^\rho}[\Gamma_{z_m}(\rho+1) - \Gamma_{z_1}(\rho+1)] \qquad 4.6$$

For stress spectra containing stress amplitudes both above and below the endurance limit s_e', the lower limit of integration should correspond to s_e' since stresses below this limit do not produce any damage. For this case z_1 should be replaced by z_e' in Eqs. 4.5 and 4.6. For the simple exponential distribution of Eq. 4.6 $z_e' = 0$ and hence Eq. 4.6 becomes $(V_R'/V_m)(e^{z_o}/z_m^\rho)\,\Gamma_{z_m}(\rho+1) = 1$

The incomplete gamma function tabulated by Pearson [10] is plotted in Figure 3. It is evident from the figure that for values of the upper limit $z > 6$ the incomplete function approaches the complete gamma function $\Gamma(\rho+1)$ very rapidly. In this region it is helpful to consider the complement of the incomplete gamma function

$$\Gamma^z(\rho+1) = \int_z^\infty z^\rho e^{-z}\,dz = \int_0^\infty z^\rho e^{-z}\,dz - \int_0^z z^\rho e^{-z}\,dz = \Gamma(\rho+1) - \Gamma_z(\rho+1) \qquad 4.7$$

Integrating the first integral of Eq. 4.7 by parts

$$\Gamma^{z}(\rho+1) = e^{-z} \sum_{n=o}^{\infty} \frac{\Gamma(\rho+1)}{\Gamma(\rho-n+1)} z^{(\rho-n)} \qquad 4.8$$

is obtained which, for integral values of ρ , may be written in the form

$$\Gamma^{z}(\rho+1) = e^{-z} \sum_{n=o}^{\rho} \frac{\rho!}{(\rho-n)!} z^{(\rho-n)} \qquad 4.9$$

Substituting Eq. 4.7 and 4.9 for instance into Eq. 4.6 the simplified form

$$\frac{V_{\dot{R}}}{V_m} \frac{e^{z_o}}{(z_m)^{\rho}} \sum_{n=o}^{\rho} \frac{\rho!}{(\rho-n)!}[z_m^{(\rho-n)} e^{-z_m} - z_1^{(\rho-n)} e^{-z_1}] = 1 \qquad 4.10$$

is obtained.

The most damaging stress amplitude s_D at the maximum rate of damage will be determined by differentiation of the damage rate $dD/ds = p(s) V_{\dot{R}}/ V'_S$ with respect to s setting the derivative equal to zero;

$$\frac{d^2D}{ds^2} = \frac{d}{ds} [\frac{V'_R}{V_m} (\frac{s-s'_e}{s_m-s'_e})^{\rho} \frac{\alpha}{s_c-s_o} (\frac{s-s_o}{s_c-s_o})^{(\alpha-1)} e^{-(\frac{s-s_o}{s_c-s_o})^{\alpha}}] = 0 \qquad 4.11$$

from which

$$(s_D-s'_e)[(\alpha-1) - \alpha (\frac{s_D-s_o}{s_c-s_o})^{\alpha}] + \rho(s_D-s_o) = 0 \qquad 4.12$$

s_D may be found given the relevant parameters. For the exponential distribution again with $\alpha = 1$, $h = 1/(s_c-s_o)$; $s_D = (\rho/h) + s'_e$.

The general S-V'_s relation, on the basis of which Eq. 4.5 was developed, is a function of the two parameters ρ and s'_e and expresses the damaging effects of the spectrum. "High level" fatigue with all stress levels considerably higher than the endurance limit is characterized by $\rho \ll \nu$ and the endurance limit remains unaffected while "low level" fatigue with all stress levels near the endurance limit, by $\rho \doteq \nu$ and $s'_e < s_e$, if the stress levels are distributed over a wide range $\rho < \nu$ and $s'_e < s_e$ will result as can be seen on Figure 3. The same relation may be useful in explaining possible work hardening effects of the high stress levels $(\rho > \nu)(s'_e > s_e)$. It is, however, expedient to keep the first parameter, ρ , constant and vary only the second one the endurance limit, s'_e. Such a procedure will permit the use of an integral value of ρ and will therefore simplify all relationships considerably. Suggestions for a constant ρ have also been made by other investigators [11] [12] , but the variation of the endurance limit was not observed until the present time.

5. Experimental Procedure and Results

Variable stress amplitude tests were performed on vertical rotating bending fatigue machines in which up to seven load levels may be applied at random to the specimen by the variation of the electric current in a coil moving in a magnetic field, the sequence of loads being controlled by a tape programming device. A detailed description of the equipment and its operation may be found in ref. 4. Three aircraft structural materials, 2024 and 7075 aluminum and SAE 4340 steel (Table 1) were tested in the form of round specimens of 5/16 in. maximum diameter and a gradually reduced 1 in. long central section of 3/16 in. minimum diameter under a great variety of stress spectra, each test series consisting of twenty specimens to permit statistical analysis of the results. A total of 1500 random and 500 constant amplitude tests were performed and their results analyzed; only the characteristic values V_R' and V_S respectively are presented here. The actual test data have been tabulated and published earlier [13, 14, 3].

The conventional S-N-L relation at the probability level $L(V_S) = 1/e$ evaluated previously without the consideration of an endurance limit has been recomputed; $\log(S-S_e)$ was plotted versus $\log V_S$ selecting S_e by trial and error in such a way as to produce a straight line. Consequently S_e is a mathematical rather than a physical endurance limit which, however, does not differ significantly from the conventional endurance limit values listed in standard tables such as ANC-5. The equations of the $(s-s_e)-V_s$ relations for the three materials are as follows:

$$\text{2024 Aluminum } V_s = 1.07 \times 10^3 \times (s - .35)^{-4.45} \tag{5.1}$$

$$\text{7075 Aluminum } V_s = 6.91 \times 10^2 \, (s - .25)^{-4.76} \tag{5.2}$$

$$\text{SAE 4340 Steel } V_s = 5.85 \times 10^2 \, (s - .46)^{-3.33} \tag{5.3}$$

and are plotted in Figure 5.

The testing machines used in the investigation can only apply discrete stress levels in random sequence rather than continuous spectra, and consequently the integration procedure of Eqs. 4.2 to 4.10 must be replaced by summation as in Eq. 2.8 where the frequency of occurrence p_i of the individual stress levels is obtained from Eq. 3.2

$$p_i = \int_{s_i}^{s_{i+1}} p(s)\,ds = e^{-\left(\frac{s_i - s_o}{s_c - s_o}\right)} - e^{-\left(\frac{s_{i+1} - s_o}{s_c - s_o}\right)} = P^*(s_i) - P^*(s_{i+1}) \tag{5.4}$$

Since stress amplitudes greater than s_m are not applied the frequency of occurrence of s_m must include those of all higher stress levels. Consequently $p_m = P^*(s_m)$. The cumulative probabilities $P^*(s)$ are tabulated in Table 2, while the stress levels used in the tests are shown in Table 3 to 5. The increment between adjacent stress levels $\Delta s = s_{i+1} - s_i$ = constant for a distribution.

Pairs of ρ and corresponding s_e' were computed by trial and error from Eq. 2.8; a few of the typical combinations are shown in Figure 6. For convenience an integer value of ρ was finally chosen for each material, s_e' was computed as the only parameter of the $(S-S_e') - V_s'$ relations and is presented in Tables 3, 4 and 5. The chosen ρ values, $\underline{\rho = 4 \text{ for aluminum and } \rho = 3 \text{ for steel}}$, provide the best fit for all tests.

The reduction of the endurance limit in random tests is quite apparent in most of the results and is most significant in the case of steel, for which such a reduction has been shown to exist [1]. Though a constant value of $\rho = 4$ produces an apparent increase of the endurance limit in a few isolated cases for 7075 aluminum (Table 5), this is only indicative of the fact that a somewhat higher value of ρ might have been chosen for these tests.

6. Conclusions

The following observations can be made on the basis of the results: (1) for 2024 aluminum and SAE 4340 steel the linear damage rule always overestimates the fatigue life as can be seen from the values of the sum of cycle ratios $1/\overline{\omega} < 1$, for 7075 aluminum the linear damage rule provides an overestimate in the majority of cases but is reliable for tests with predominantly very low stresses; (2) a constant value of ρ may be found for each material; this and a variable endurance limit stress will determine the interaction damage $(S-S_e') - V_s'$ diagram, permitting the use of a quasi-linear damage rule; (3) empirical relationships between s_e' and the other relevant variables, namely, h, s_1, s_m, V_R, and V_m, may be determined at least for 2024 aluminum and SAE 4340 steel; they give a fairly reliable estimate of the lowered endurance limit (for constant ρ) as demonstrated in Figures 7 and 8 and Eqs. 5.5 and 5.6.

$$\text{2024 Aluminum } s_e' = .145 \log h^2 s_1 \left(\frac{V_R}{V_m}\right)^{1/4} -.18 \qquad 5.5$$

$$\text{SAE 4340 Steel } s_e' = s_m - .57 \left(\frac{V_R}{V_m}\right)^{1/5} \left(\frac{h}{\alpha}\right)^{1/2} s_1^4 -.352 \qquad 5.6$$

No such a relation was however found for 7075 Aluminum.

It is apparent that the constant value ρ with $s_e' = 0$ provides a safe fatigue life for all tests, while a careful choice of the endurance limit reduced by about 25% will give conservative estimates in most cases. The constant values of ρ are only slightly lower than the conventional slopes ν of the $\log(\bar{S}-S_e) - \log V_S$ diagrams; as a matter of fact they are the nearest integer values to ν and suggest that similar procedures may be followed for other materials. The approximate value of the reduced endurance limit may then be obtained from a few program tests since s_e' is delimited by zero on the one hand and the conventional endurance limit s_e on the other.

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TABLE 1 PHYSICAL PROPERTIES OF MATERIALS

	Ultimate Tensile Strength σ_u ksi	Yield Strength in Tension σ_y ksi	Modulus of Elasticity $E \times 10^{-6}$ ksi	Slope ν of $\log(S-S_e)$-$\log V_s$ line	Endurance Limit Stress Ratio s_e	Slope ρ of $\log(S-S_e')$-$\log V_s'$ line
2024 Aluminum	64	53	10	4.46	.35	4
7075 Aluminum	82	66	10	4.76	.25	4
SAE 4340 Steel	140	130	30	3.33	.46	3

TABLE 2 PARAMETERS OF LOAD DISTRIBUTIONS

Dis-tri-but-ion	s_o	s_c	α	Frequencies of Occurrence $P^*(s_i)$ of Stress Amplitude Ratios Equal to or exceeding s_i $P^*(s_1)$	$P^*(s_2)$	$P^*(s_3)$	$P^*(s_4)$	$P^*(s_5)$	$P^*(s_6)$
A	$s_1-.2\Delta s$	$s_1+1.8\Delta s$	2.0	1.0	.95000	.45000	.10000	.030000	.01000000
B	$s_1-.2\Delta s$	$s_1+2.5\Delta s$	2.5	1.0	.98000	.80000	.30000	.050000	.01000000
C	$s_1-.2\Delta s$	$s_1+1.3\Delta s$	1.0	1.0	.50000	.25000	.12000	.050000	.01000000
D	s_1	$s_1+.6\Delta s$	1.0	1.0	.19406	.05302	.01272	.002670	.00066000
A'	$s_1-.3\Delta s$	$s_1+1.8\Delta s$	2.1	1.0	.97000	.34500	.04500	.007000	.00200000
B'	$s_1-.2\Delta s$	$s_1+2.5\Delta s$	2.6	1.0	.98500	.88700	.26200	.012000	.00200000
C'	$s_1-.3\Delta s$	$s_1+.7\Delta s$	1.0	1.0	.37500	.12500	.04200	.012000	.00200000
A"	$s_1-.1\Delta s$	$s_1+2.2\Delta s$	1.6	1.0	.90000	.50000	.22000	.110000	.05000000
B"	$s_1-.4\Delta s$	$s_1+2.7\Delta s$	1.8	1.0	.94000	.76000	.36000	.130000	.05000000
C"	$s_1-.1\Delta s$	$s_1+1.7\Delta s$	1.0	1.0	.60000	.37000	.23000	.130000	.05000000
E	s_1	$s_1+.578\Delta s$	1.0	1.0	.17800	.03240	.00576	.001180	.00018000
F	s_1	$s_1+.437\Delta s$	1.0	1.0	.10000	.01000	.00100	.000100	.00001000
G	s_1	$s_1+.292\Delta s$	1.0	1.0	.03160	.00100	.00003	.000001	.00000003

TABLE 3 PARAMETERS AND TEST RESULTS FOR 2024 ALUMINUM SPECIMENS

$\rho = 4.$, $\nu = 4.46$, $s_e = .35$

Test Series No.	Spectrum Type (Table II)	Lowest Stress Amplitude Ratio s_1	Stress Ratio Increment Δs	No. of Levels in Spectrum n	Linear Life (MINER) V_R in Thousands of Cycles	Test Results; Fatigue Life V_R' in Thousands of Cycles	Endurance Limit Ratio s_e'	Cumulative Cycle Ratio $1/\bar{\omega}$
1	A	.372	.0970	6	608.0	166.6	.131	.274
2	B	.372	.0970	6	325.0	109.5	.153	.334
3	C	.372	.0970	6	611.0	150.5	.014	.246
4	A	.390	.1015	6	405.0	134.1	.172	.332
5	B	.390	.1015	6	217.0	74.1	.137	.341
6	C	.390	.1015	6	408.0	119.6	.053	.293
7	D	.390	.1015	6	3,620.0	495.5	.160	.137
8	A'	.390	.1015	6	790.0	134.8	.102	.171
9	B'	.390	.1015	6	275.0	103.8	.216	.377
10	C'	.390	.1015	6	1,090.0	203.4	.054	.187
11	A"	.390	.1015	6	163.0	56.5	0	.347
12	B"	.390	.1015	6	129.0	45.8	0	.355
13	C"	.390	.1015	6	161.0	62.8	0	.390
14	A	.441	.0508	6	1,580.0	306.0	0	.194
15	B	.441	.0508	6	866.0	180.0	0	.208
16	C	.441	.0508	6	1,870.0	285.0	0	.152
17	D	.289	.1015	6	16,950.0	6,523.0	.200	.385
18	C"	.289	.1015	6	489.0	132.7	0	.271
19	A	.645	.1015	6	83.1	49.7	.310	.598
20	B	.645	.1015	6	59.9	37.6	.305	.628
21	C	.645	.1015	6	103.0	51.4	.169	.499
22	E	.350	.1000	6	14,120.0	3,760.0	.233	.266
23	E	.450	.1000	6	2,510.0	479.0	.210	.191
24	E	.550	.1000	5	492.0	81.8	.286	.166
25	E	.650	.1000	4	138.0	53.3	.160	.386
26	F	.350	.1000	6	48,120.0	13,308.0	.263	.277
27	F	.450	.1000	6	5,931.0	1,420.0	.275	.239
28	F	.550	.1000	5	733.0	259.0	.222	.353
29	F	.650	.1000	4	174.0	71.3	.217	.410
30	G	.450	.1000	6	15,100.0	4,400.0	.314	.291
31	G	.550	.1000	5	996.0	477.0	.310	.479
32	G	.650	.1000	4	209.0	116.0	.300	.555

TABLE 4 PARAMETERS AND TEST RESULTS FOR 7075 ALUMINUM SPECIMENS

$\rho = 4$, $\gamma = 4.76$, $s_e = .25$

Test Series No.	Spectrum Type (Table II)	Lowest Stress Amplitude Ratio s_1	Stress Ratio Increment Δs	No. of Levels in Spectrum n	Linear Life (Miner) V_R in Thousands of Cycles	Test Results; Fatigue Life V'_R in Thousands of Cycles	Endurance Limit Ratio s'_e	Cumulative Cycle Ratio $1/\bar{\omega}$
1	A	.360	.094	6	196.0	54.5	0	.278
2	B	.360	.094	6	110.2	38.5	0	.349
3	C	.360	.094	6	201.4	143.8	.24	.714
4	D	.360	.094	6	1,389.8	600.0	.20	.432
5	A'	.360	.094	6	353.0	201.0	.27	.569
6	B'	.360	.094	6	140.3	97.0	.32	.693
7	C'	.360	.094	6	554.5	230.5	.15	.416
8	A"	.360	.094	6	82.2	33.9	0	.412
9	B"	.360	.094	6	65.8	29.5	0	.448
10	C"	.360	.094	6	81.6	73.1	.25	.896
11	A	.313	.047	6	3,004.5	2,460.0	.27	.819
12	B	.313	.047	6	1,852.1	1,970.0	.40	1.064
13	C	.313	.047	6	3,572.8	3,674.0	.34	1.028
14	D	.313	.047	6	12,315.0	4,319.9	.12	.351
15	A	.595	.047	6	42.8	27.9	.17	.652
16	B	.595	.047	6	31.0	19.5	.06	.629
17	C	.595	.047	6	53.0	33.2	.18	.626
18	C'	.266	.094	6	2,092.2	1,282.0	.22	.613
19	C"	.266	.094	6	244.6	192.3	.21	.786
20	E	.350	.100	6	2,099.0	694.3	.17	.331
21	E	.450	.100	5	452.6	197.5	.22	.436
22	E	.550	.100	4	118.1	50.8	.18	.430
23	F	.350	.100	6	4,413.8	1,467.4	.20	.332
24	F	.450	.100	5	748.5	220.2	.20	.294
25	F	.550	.100	4	166.6	52.3	.14	.314
26	G	.350	.100	6	7,686.4	9,493.0	.28	1.235
27	G	.450	.100	5	1,063.6	336.1	.21	.316
28	G	.550	.100	4	213.0	92.0	.22	.432

TABLE 5 PARAMETERS AND TEST RESULTS FOR SAE 4340 STEEL SPECIMENS

$\rho = 3, \quad \nu = 3.33, \quad s_e = .46$

Test Series No.	Spectrum Type (Table II)	Lowest Stress Amplitude Ratio s_1	Stress Ratio Increments Δs	No. of Stress Levels in Spectrum n	Linear Life (Miner) V_R in Thousands of Cycles	Test Results Fatigue Life V'_R in Thousands of Cycles	Endurance Limit Ratio s'_e	Cumulative Cycle Ratio $1/\bar{\omega}$
1	A'	.514	.0714	6	195.6	72.6	.343	.371
2	B'	.514	.0714	6	91.4	28.3	.092	.310
3	C'	.514	.0714	6	380.0	64.0	.156	.168
4	D	.514	.0714	6	954.0	133.9	.276	.140
5	C	.443	.0714	6	1,363.0	279.0	.214	.205
6	D	.443	.0714	6	4,178.0	796.0	.380	.191
7	E	.350	.1000	6	9,330.0	1,550.0	.288	.166
8	E	.450	.1000	5	2,054.0	267.0	.263	.130
9	E	.550	.1000	4	320.4	168.0	.379	.524
10	F	.350	.1000	6	49,220.0	3,480.0	.292	.071
11	F	.450	.1000	5	4,945.0	497.0	.315	.101
12	F	.550	.1000	4	494.5	235.0	.389	.475
13	G	.350	.1000	6	709,220.0	70,000.0*	.400	.100
14	G	.450	.1000	5	22,334.0	2,180.0	.336	.098
15	G	.550	.1000	4	705.7	330.0	.404	.468

* Estimated Value

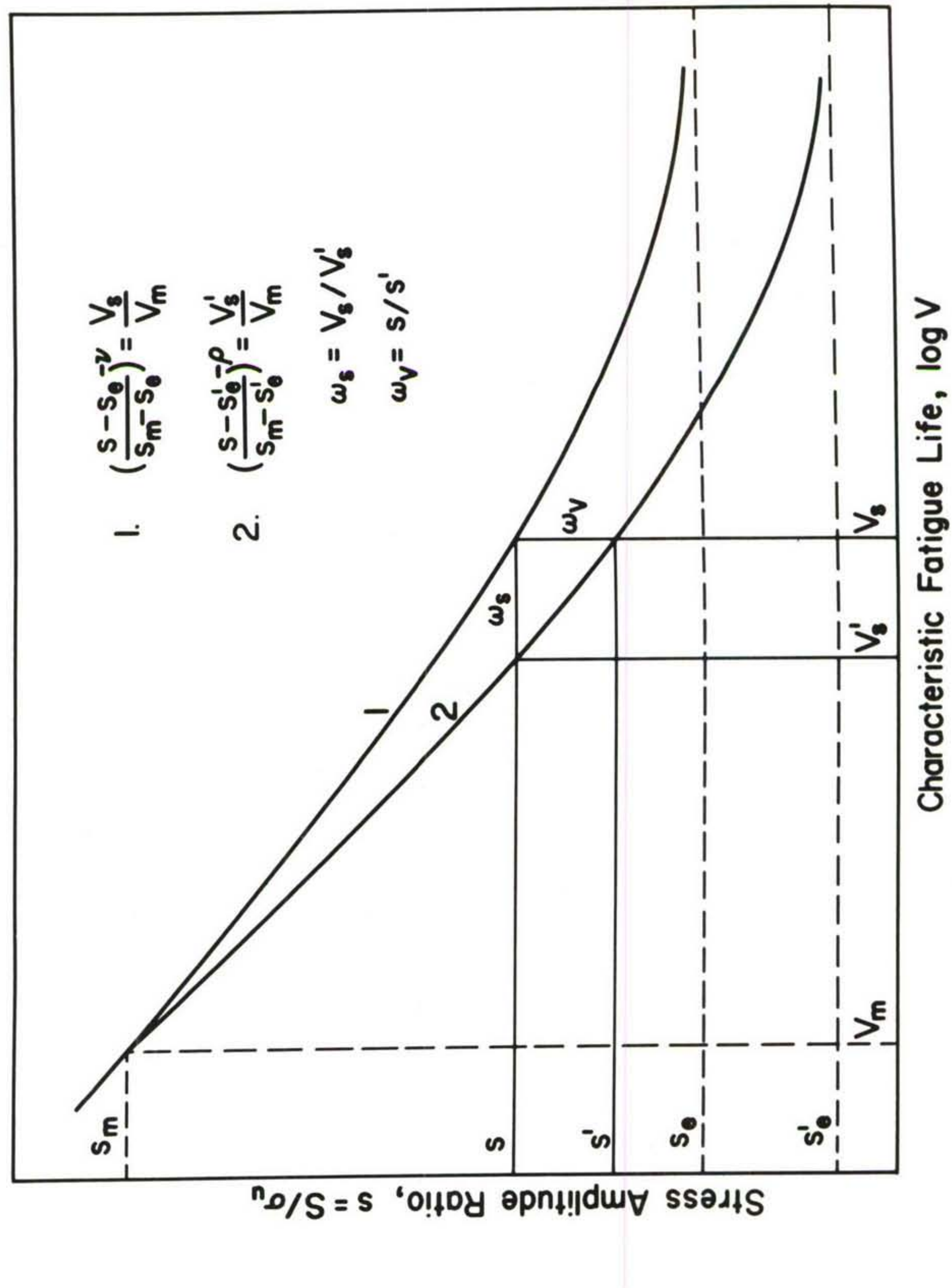

FIGURE 1 . CONVENTIONAL $S - V_s$ AND INTERACTION DAMAGE DIAGRAM

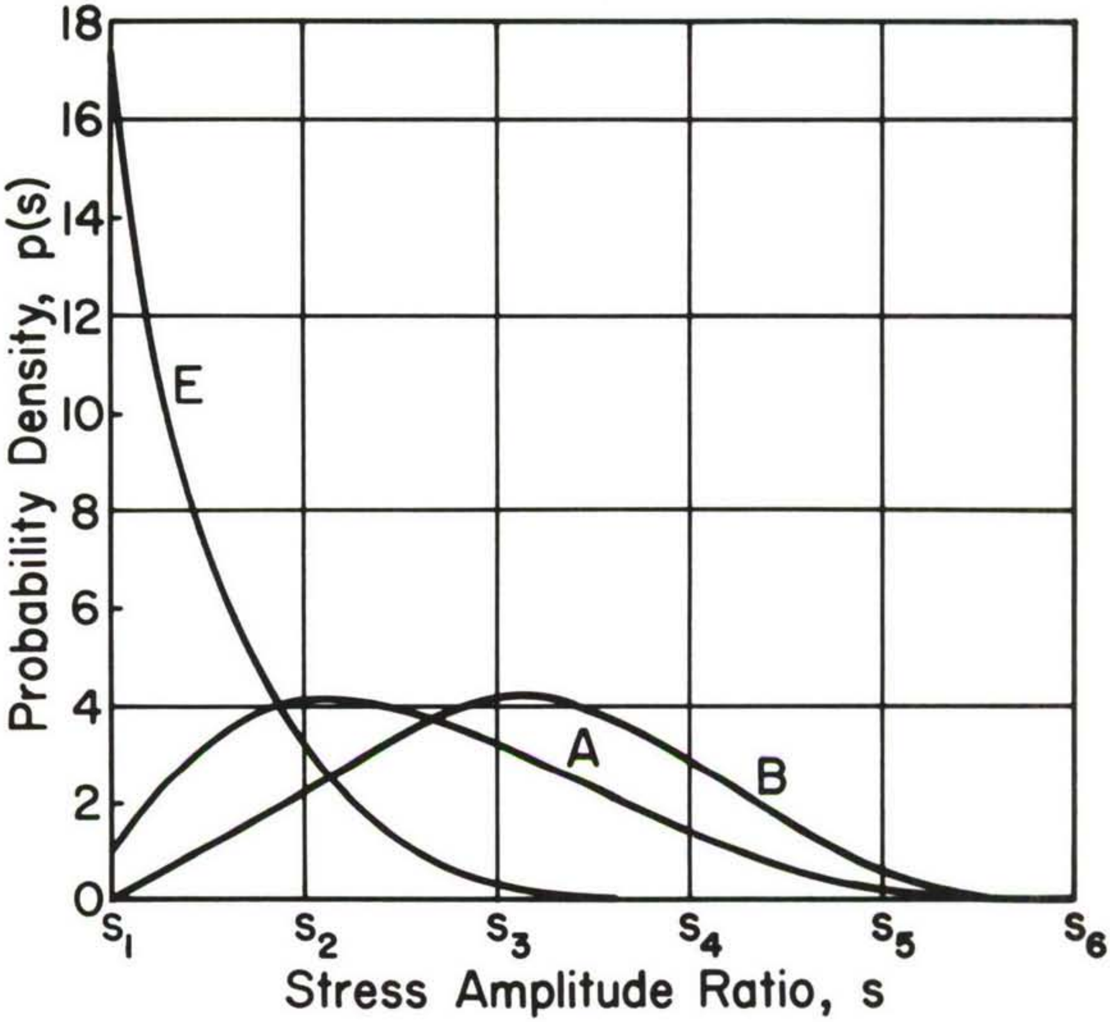

FIGURE 2. TYPICAL LOAD SPECTRA

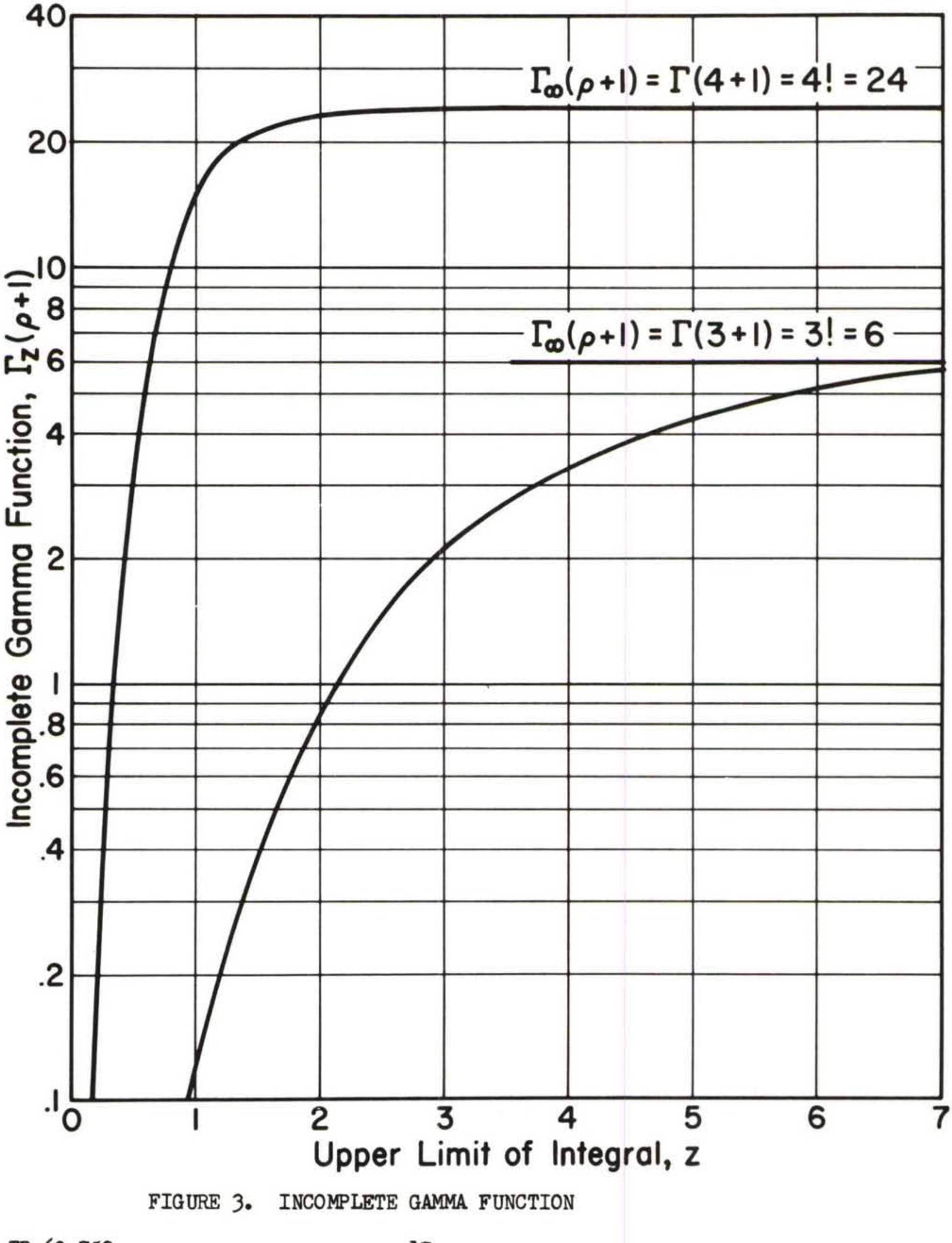

FIGURE 3. INCOMPLETE GAMMA FUNCTION

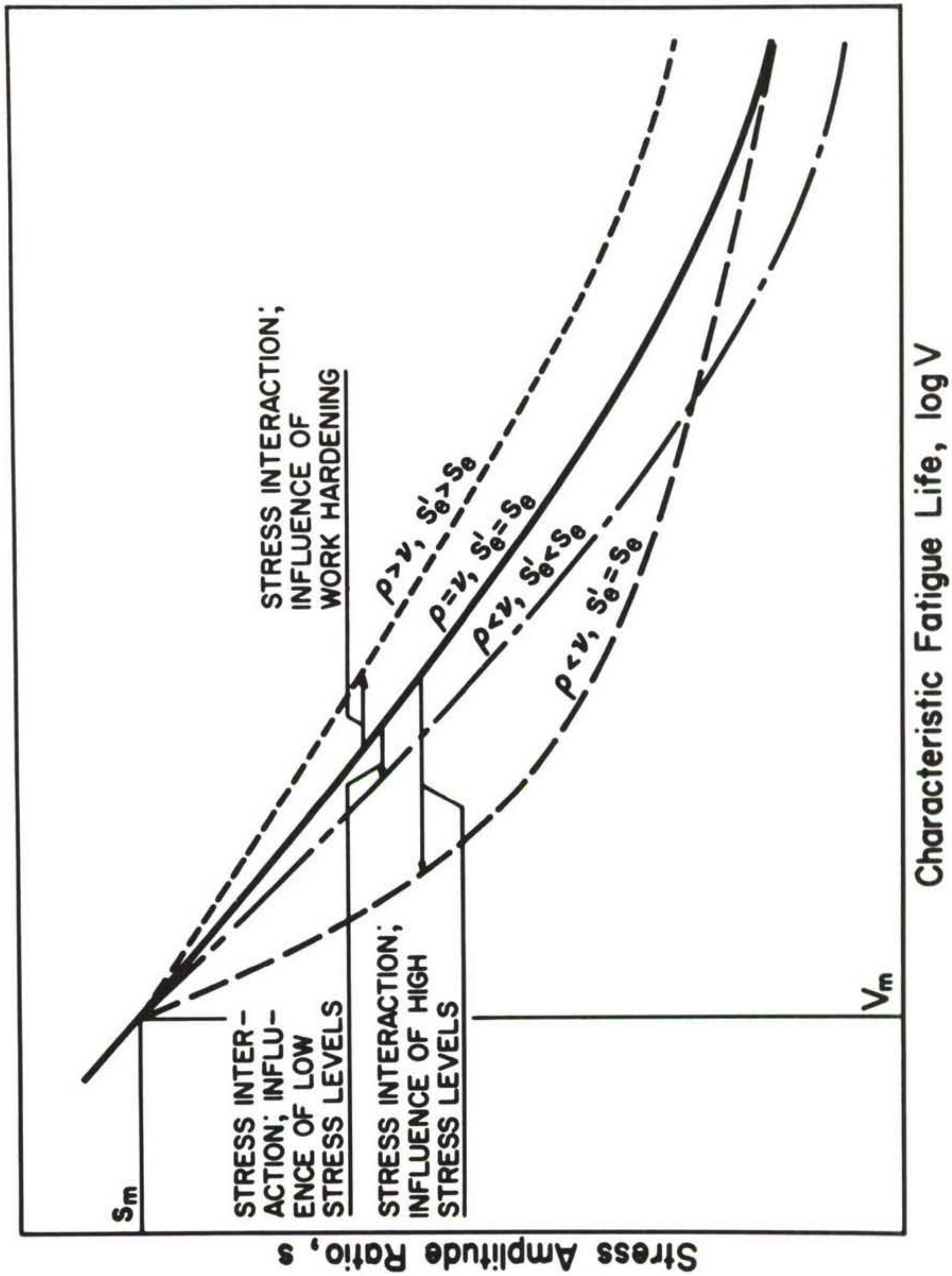

FIGURE 4. VARIOUS TYPES OF INTERACTION DIAGRAMS

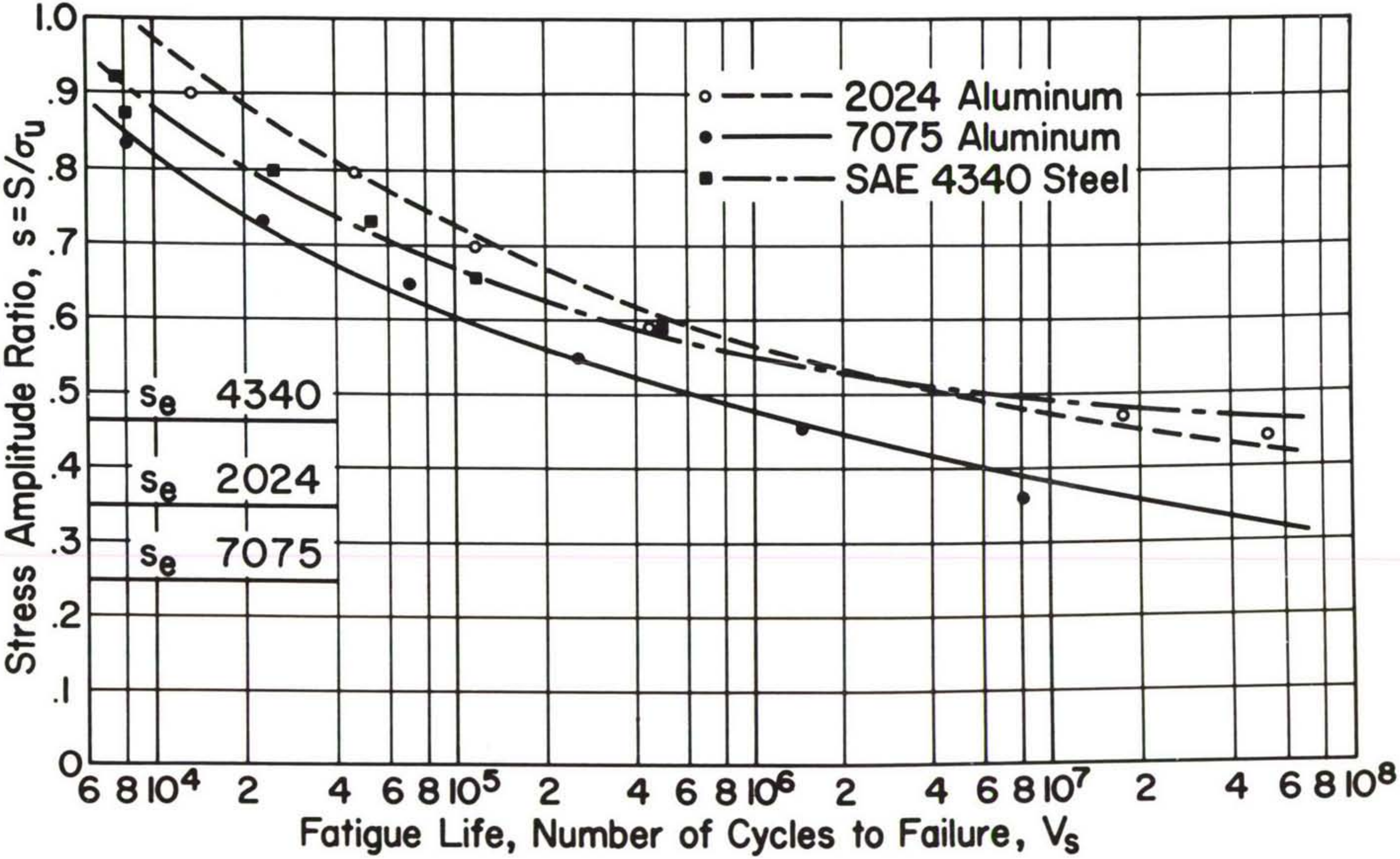

FIGURE 5. CONSTANT AMPLITUDE FATIGUE DIAGRAMS FOR THE INVESTIGATED MATERIALS

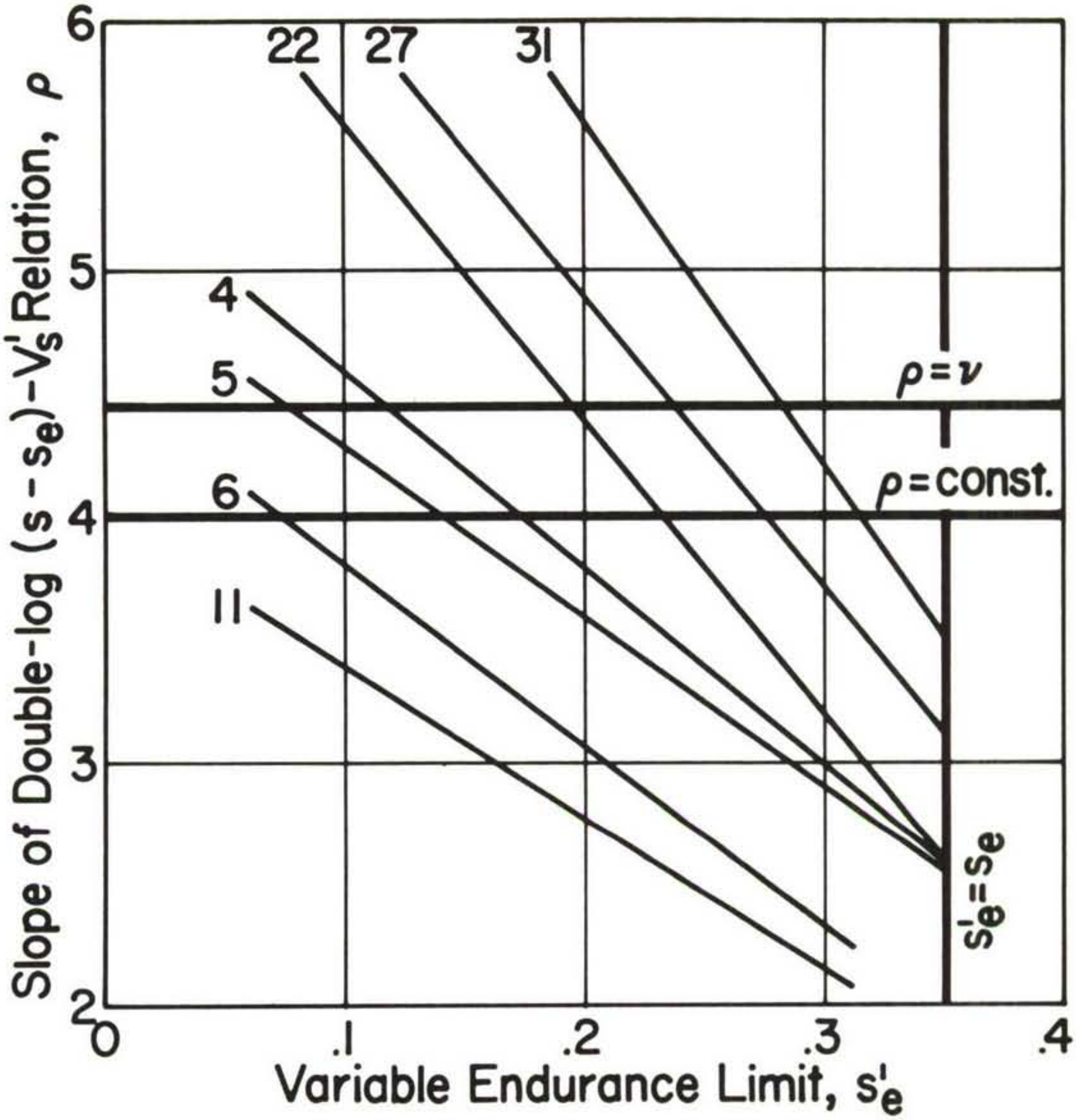

FIGURE 6. VARIATION OF ρ WITH s_e' AND RELEVANT VARIABLES FOR 2024 ALUMINUM

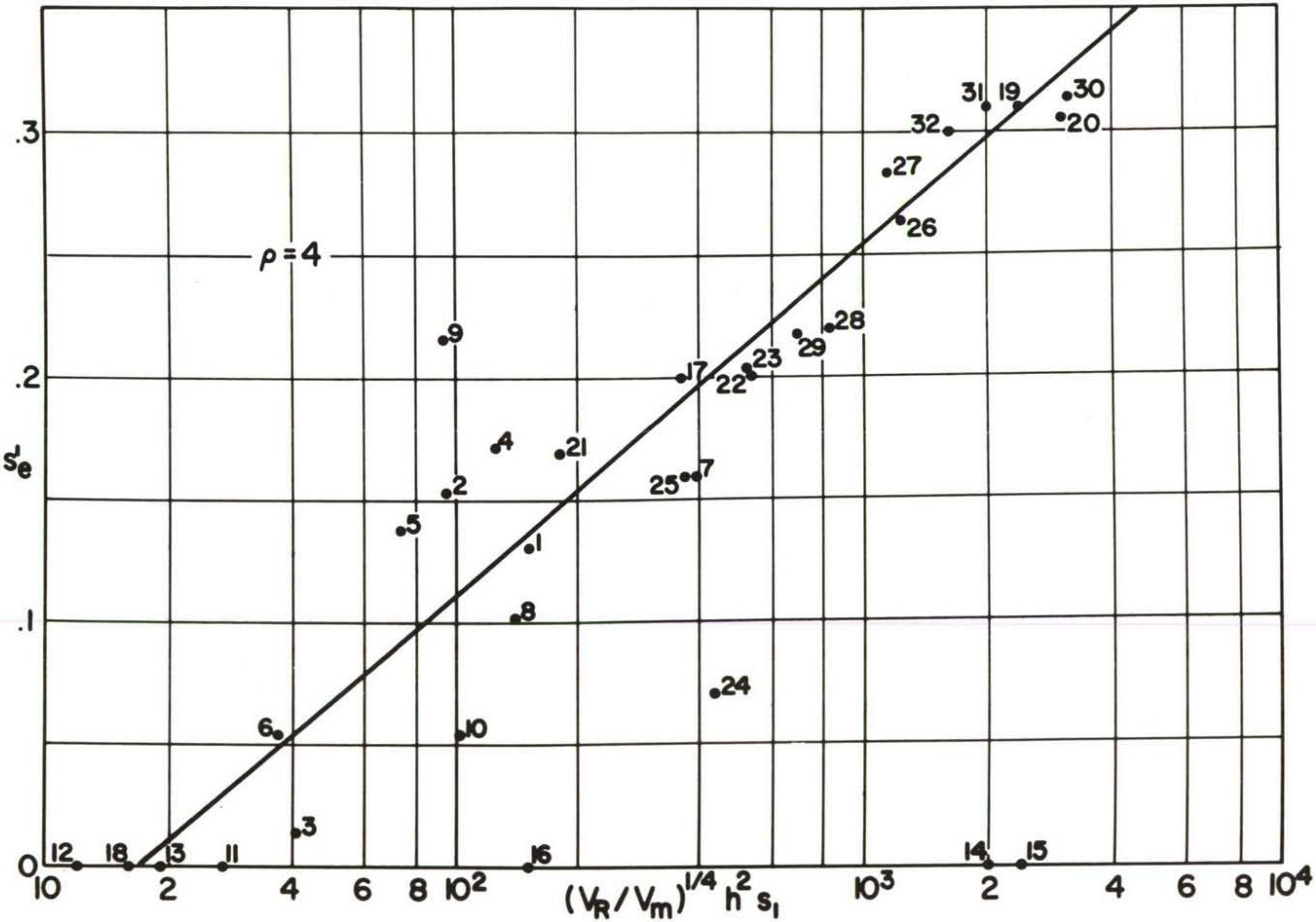

FIGURE 7. EMPIRICAL RELATION BETWEEN s'_e AND RELEVANT VARIABLES FOR 2024 ALUMINUM

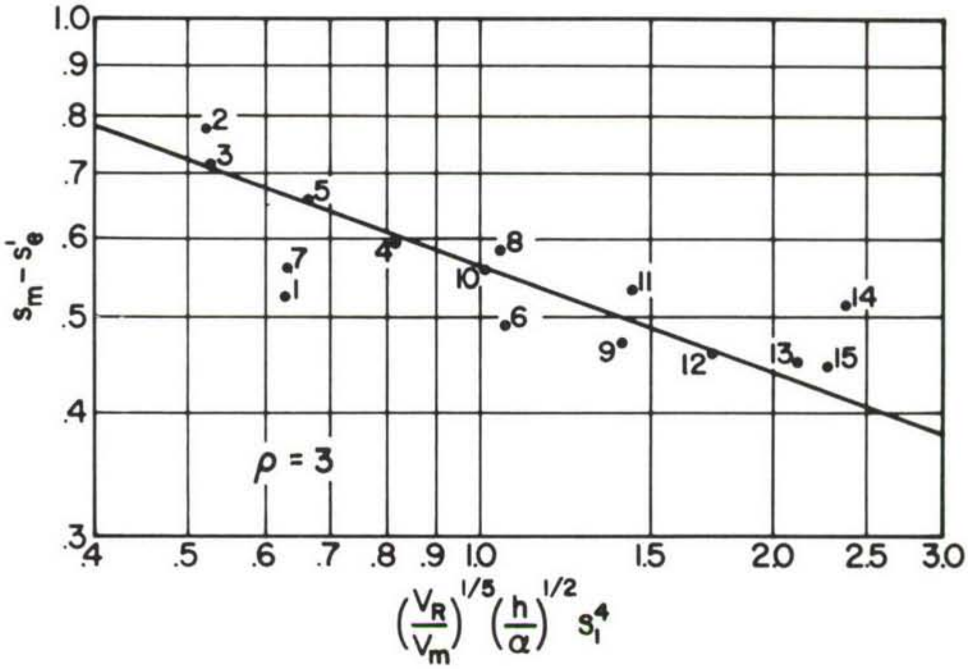

FIGURE 8. EMPIRICAL RELATION BETWEEN s_e' AND RELEVANT VARIABLES FOR SAE 4340 STEEL

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(Contract AF 33(616)-7042)
Unclassified report

Results of an investigation of the effects of stress interaction on the fatigue life of aircraft structural materials subjected to randomized load spectra. All three materials 2024 and 7075 aluminum and SAE 4340 steel exhibit fatigue lives shorter than those predicted on the basis of the linear

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(Miner) damage rule. A quasi-liner rule is proposed with a variable, spectrum dependent, endurance limit producing safe life estimates; the dependence of the endurance limit on the stress spectrum and its resulting design inadequacy is shown. Tests were performed on high speed, programmed, rotating bending fatigue machines of special design.

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